# Cancellation of nonrenormalizable hypersurface divergences and the $d$-dimensional Casimir piston

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**Abstract**

Using a multidimensional cut-off technique, we obtain expressions for the cut-off dependent part of the vacuum energy for parallelepiped geometries in any spatial dimension $d$. The cut-off part yields nonrenormalizable hypersurface divergences and we show explicitly that they cancel in the Casimir piston scenario in all dimensions. We obtain two different expressions for the $d$-dimensional Casimir force on the piston where one expression is more convenient to use when the plate separation $a$ is large and the other when $a$ is small (a useful $a \rightarrow 1/a$ duality). The Casimir force on the piston is found to be attractive (negative) for any dimension $d$. We apply the $d$-dimensional formulas (both expressions) to the two and three-dimensional Casimir piston with Neumann boundary conditions. The 3D Neumann results are in numerical agreement with those recently derived in arXiv:0705.0139 using an optical path technique providing an independent confirmation of our multidimensional approach. We limit our study to massless scalar fields.

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# 1   Introduction

With idealized boundary conditions, such as Dirichlet and Neumann boundary conditions, vacuum energy calculations have cut-off dependent terms that diverge in the limit as the cut-off $\Lambda$ tends to infinity. These divergences can be classified as either bulk (volume) divergences or lower-dimensional surface divergences (for simplicity, all divergences besides the volume divergence will be classified as hypersurface or simply surface divergences regardless of the dimension). The volume divergence poses no problems since the Casimir energy, by definition, is obtained after subtracting out the volume divergence from the vacuum energy. In other words, the volume divergence is renormalizable and can be set to zero by simply adding a constant counterterm to the Hamiltonian. In contrast, the surface divergences are nonrenormalizable. It is tempting to throw out the surface divergence as an artifact of idealized boundary conditions and retain the finite part as the true Casimir energy as is often done in the literature (see references in [4]). However, this is not a physically valid renormalization procedure. It has been shown that these surface divergences cannot be removed via renormalization of any physical parameters of the theory [1, 2, 3]. In the zeta- function regularization technique, these surface divergences do not appear because in effect they are renormalized to zero[1]. In special cases, like the parallel plate geometry with infinite plates, this is not an issue because in the limit as $\Lambda \to \infty$ the Casimir force is finite. However, in any realistic situation where the plates are of finite size, the Casimir force diverges in the limit $\Lambda \to \infty$.

Unambiguous Casimir calculations can be carried out with idealized boundary conditions in the apparatus called the Casimir piston. A few years ago, Cavalcanti [5] showed for the case of a two-dimensional (2+1) massless scalar field confined to a rectangular region with Dirichlet boundary conditions that the Casimir piston can resolve the issue of nonrenormalizable surface divergences that appear in Casimir calculations. A Casimir piston contains an interior and an exterior region and Cavalcanti showed explicitly that the surface cut-off terms of the interior and exterior regions canceled. He also showed that the Casimir force on the piston is always negative regardless of the ratios of the two sides of the rectangular region. This is in contrast to calculations that can yield positive Casimir forces in rectangular geometries when the surface cut-off terms are thrown out and no exterior region is considered (see references in [4]).

In this paper, we use a multidimensional cut-off technique [6] to obtain exact expressions for the cut-off dependent ($\Lambda$-dependent) part of the Casimir energy for a $d$-dimensional parallelepiped region. In the limit $\Lambda \to \infty$, these yield nonrenormalizable hypersurface divergences and we show explicitly that they cancel out in the Casimir piston scenario for any dimension. We then derive exact expressions for the Casimir force on the piston in any dimension $d$ and use the invariance of the vacuum energy under permutations of lengths to derive an alternative

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[1]Like dimensional regularization, zeta-function regularization goes beyond pure regularization and does some renormalization.

expression. When the plate separation $a$ is large, an otherwise long computation using the first expression becomes trivial using the alternative expression and vice versa when $a$ is small (there is a useful $a \to 1/a$ duality). As in two and three dimensions, the Casimir force on the piston is attractive (negative) in any spatial dimension $d$.

For the three-dimensional Casimir piston with massless scalar fields obeying Dirichlet and Neumann boundary conditions approximate results were first obtained in [7] for small plate separation. Exact results for arbitrary plate separation were then obtained for the Dirichlet case in [8]. Exact results for the 3D Neumann (as well as Dirichlet) case were recently obtained via an optical path technique [12]. In [12], arbitrary cross sections, temperature and free energy were also studied. We apply our $d$-dimensional formulas (both expressions) to the 2D and 3D Neumann cases. The first 3D expression looks similar in form to the one recently derived in [12] and is in numerical agreement with it. The alternative 3D expression converges quickly when $a$ is large and though it is quite different in form compared to the first expression or the one found in [12] it is in numerical agreement with both of them. The 2D Neumann results are new and bring a completeness to the original work of Cavalcanti [5] where the 2D Dirichlet case was considered. Before discussing the literature on Casimir pistons for the electromagnetic case, it is worth noting that the use of massless scalar fields in Casimir studies goes beyond theoretical interest and has direct application to physical systems such as Bose-Einstein condensates [20, 21, 22]. Higher-dimensional scalar field Casimir calculations have also been carried out in the context of 6D supergravity theories [23].

For perfect-conductor conditions, the Casimir piston for the electromagnetic field in a three-dimensional rectangular cavity was studied in [7] and the Casimir force on the piston was found to be attractive in contrast to results without exterior region where the force could be positive. This was then generalized further in Refs. [9]-[12] where the temperature and free energy dependence was studied. It was shown in [13] that the Casimir force between two bodies related by reflection is always attractive, independent of the exact form of the bodies or dielectric properties and this was generalized further in [14]. It has also been shown that Casimir piston scenarios can yield repulsive forces. The Casimir piston for a weakly reflecting dielectric was considered in [15] and it was shown that though attraction occurred for small plate separation, this could switch to repulsion for sufficiently large separation. However, the force remained attractive for all plate separations if the material was thick enough, in agreement with the results in [7]. Two preprints [16, 17] also discuss scenarios where repulsive Casimir forces in pistons can be achieved. Recently, two independent groups have developed techniques for calculating Casimir forces between arbitrary compact objects and have applied the results to the case of two spherical bodies at a distance [18, 19].

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# 2    Cancellation of hypersurface divergences in the $d$-dimensional Casimir piston

The expression for the vacuum energy $\tilde{E}$ regularized using a multidimensional cut-off technique [6] divides naturally into two parts: a finite part $E_0$ and a cut-off dependent part $E(\Lambda)$ which diverges as the cut-off $\Lambda$ tends to infinity. The expressions for $E(\Lambda)$ and $E_0$ are derived in appendix A and are written in a compact fashion with the help of the *ordered* symbol $\xi^d_{k_1...k_j}$ which was introduced in [8] and is defined below after the expressions for $E(\Lambda)$ are written. Our goal in this section is to show that for the Casimir piston scenario, the hypersurface divergences of the interior and exterior regions of the piston cancel out for any dimension $d$. By "cancel out" we do not mean that the cut-off dependent part of the Casimir energy is zero but that it is independent of the plate separation $a$ so that the Casimir *force* on the piston has no cut-off dependence. In the next section we focus on the finite part $E_0$ and obtain explicit expressions for the Casimir force on the piston in any dimension. We work in units where $\hbar = c = 1$.

The regularized vacuum energy for massless scalar fields in a $d$-dimensional box with sides of arbitrary lengths $L_1, L_2, ..., L_d$ obeying Dirichlet (D) boundary conditions is given by (A.25) (as $\Lambda \to \infty$)

$$\tilde{E}_D = E_{0_D} + E_D(\Lambda) \tag{2.1}$$

where $E_{0_D}$ is the finite part for the Dirichlet case and $E_D(\Lambda)$ is the cut-off dependent part given by (A.13):

$$
\begin{aligned}
E_D(\Lambda) &\equiv \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m} \, m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\tfrac{m+1}{2}) \, \Lambda^{m+1} \, \xi^d_{k_1,..,k_m} \prod_{i=1}^{m} L_{k_i} \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m} \, m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\tfrac{m+1}{2}) \, \Lambda^{m+1} \, \xi^d_{k_1,..,k_m} (L_{k_1} \ldots L_{k_m}).
\end{aligned}
\tag{2.2}
$$

For the case of Neumann(N) boundary conditions the regularized vacuum energy is given by (A.21) (as $\Lambda \to \infty$)

$$\tilde{E}_N = E_{0_N} + E_N(\Lambda) \tag{2.3}$$

where $E_{0_N}$ is the finite part and $E_N(\Lambda)$ is the cut-off dependent part given by (A.14):

$$
\begin{aligned}
E_N(\Lambda) &\equiv \frac{1}{2^{d+1}} \sum_{m=1}^{d} m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\tfrac{m+1}{2}) \, \Lambda^{m+1} \, \xi^d_{k_1,..,k_m} \prod_{i=1}^{m} L_{k_i} \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\tfrac{m+1}{2}) \, \Lambda^{m+1} \, \xi^d_{k_1,..,k_m} (L_{k_1} \ldots L_{k_m}).
\end{aligned}
\tag{2.4}
$$

In this section, the expressions for $E_{0_D}$ and $E_{0_N}$ are not needed.

There is an implicit summation over the integers $k_i$ in (2.2) and (2.4). The ordered symbol $\xi^d_{k_1,..,k_m}$ [8] is defined by

$$\xi^d_{k_1,..,k_m} = \begin{cases} 1 & \text{if } k_1 < k_2 < \ldots < k_m \,;\, 1 \le k_m \le d \\ 0 & \text{otherwise .} \end{cases} \tag{2.5}$$

The ordered symbol ensures that the implicit sum over the $k_i$ is over all distinct sets $\{k_1, \ldots, k_m\}$, where the $k_i$ are integers that can run from 1 to $d$ inclusively under the constraint that $k_1 < k_2 < \cdots < k_m$. The superscript $d$ specifies the maximum value of $k_m$. For example, if $m = 2$ and $d = 3$ then $\xi^d_{k_1,..,k_m} = \xi^3_{k_1,k_2}$ and the non-zero terms are $\xi_{1,2}$, $\xi_{1,3}$ and $\xi_{2,3}$. This means the summation is over $\{k_1, k_2\} = (1,2), (1,3)$ and $(2,3)$ so that $\xi^3_{k_1,k_2} L_{k_1} L_{k_2} = L_1 L_2 + L_1 L_3 + L_2 L_3$.

## 2.1 Cancellation in three dimensions

Before showing how the cut-off dependent hypersurface divergences in the $d$-dimensional Casimir piston cancel, we consider the case of three spatial dimensions first. Three dimensions allows us to make the first non-trivial use of the $d$-dimensional cut-off expressions (2.2) and (2.4) and to illustrate in a transparent fashion how the cancellation occurs. This paves the way to follow the cancellation in $d$-dimensions in the next subsection. The cut-off expressions in three dimensions and their cancellation in the piston scenario are in agreement with the work in [7, 12] and this provides an independent confirmation of our general formulas.

In three dimensions, the Dirichlet and Neumann cut-off expressions $E_D(\Lambda)$ and $E_N(\Lambda)$, are obtained by substituting $d = 3$ in equations (2.2) and (2.4):

$$E_D(\Lambda) = \frac{1}{2^4} \sum_{m=1}^{3} (-1)^{3+m} \, m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\tfrac{m+1}{2}) \, \Lambda^{m+1} \, \xi^3_{k_1,..,k_m} \, (L_{k_1} \ldots L_{k_m})$$
$$= \frac{\pi}{8} \Lambda^2 (L_1 + L_2 + L_3) - \frac{\pi^2}{4} \Lambda^3 (L_1 L_2 + L_1 L_3 + L_2 L_3) + \frac{3\pi^2}{2} \Lambda^4 L_1 L_2 L_3 \tag{2.6}$$

$$E_N(\Lambda) = \frac{1}{2^4} \sum_{m=1}^{3} \xi^3_{k_1,..,k_m} \, (L_{k_1} \ldots L_{k_m}) \, m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\tfrac{m+1}{2}) \, \Lambda^{m+1}$$
$$= \frac{\pi}{8} \Lambda^2 (L_1 + L_2 + L_3) + \frac{\pi^2}{4} \Lambda^3 (L_1 L_2 + L_1 L_3 + L_2 L_3) + \frac{3\pi^2}{2} \Lambda^4 L_1 L_2 L_3 \,. \tag{2.7}$$

Except for a trivial redefinition of $\Lambda \to \Lambda/\pi$, the above cut-off expressions in three dimensions are in agreement with those derived in [12]. The $\Lambda^4$ term appearing in (2.6) and (2.7) is
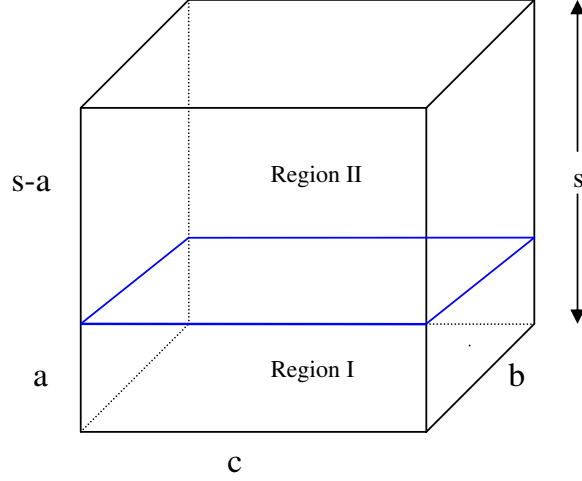
Figure 1: Casimir piston in three dimensions.

multiplied by the volume $L_1 L_2 L_3$ of the box and represent the volume divergence of the continuum. This volume term poses no divergence problems since it must be subtracted to obtain the Casimir energy (defined as the difference between the vacuum energy with boundaries and the bulk vacuum energy of the continuum). In other words, it can be renormalized to zero via a constant counterterm in the Hamiltonian. The remaining $\Lambda^2$ and $\Lambda^3$ terms are proportional to the perimeter and surface area respectively (we refer to both as surface divergences for simplicity). In contrast to the volume divergence, there is no physical justification for subtracting out the surface divergences. In other words, they cannot be renormalized to zero.

For the Casimir piston, the Casimir energy is obtained by adding the vacuum energy of the interior region I and exterior region II (see fig. 1). To obtain the cut-off dependence for the Casimir energy we therefore add the cut-off terms (the surface divergences) in regions I and II. Let the plate separation be $a$. In region I, the three lengths are $L_1 = a$, $L_2 = b$ and $L_3 = c$ whereas in region II the three lengths are $L_1 = s - a$, $L_2 = b$ and $L_3 = c$. Note that in region I, $L_1$ comes with $+a$ whereas in region II, $L_1$ comes with the opposite sign $-a$. For Dirichlet

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boundary conditions we obtain

$$
\begin{aligned}
E_{D_{piston}}(\Lambda) &= E_{D_1}(\Lambda) + E_{D_2}(\Lambda) \\
&= \frac{\pi}{8}\Lambda^2(a+b+c) - \frac{\pi^2}{4}\Lambda^3(a\,b + a\,c + b\,c) \\
&\quad + \frac{\pi}{8}\Lambda^2\big(s - a + b + c\big) - \frac{\pi^2}{4}\Lambda^3\big((s-a)\,b + (s-a)\,c + b\,c\big) \\
&= \frac{\pi}{8}\Lambda^2(s + 2\,b + 2\,c) - \frac{\pi^2}{4}\Lambda^3(s\,b + s\,c + 2\,b\,c)
\end{aligned}
\tag{2.8}
$$

For Neumann boundary conditions we obtain

$$
\begin{aligned}
E_{N_{piston}}(\Lambda) &= E_{N_1}(\Lambda) + E_{N_2}(\Lambda) \\
&= \frac{\pi}{8}\Lambda^2(a+b+c) + \frac{\pi^2}{4}\Lambda^3(a\,b + a\,c + b\,c) \\
&\quad + \frac{\pi}{8}\Lambda^2\big(s - a + b + c\big) + \frac{\pi^2}{4}\Lambda^3\big((s-a)\,b + (s-a)\,c + b\,c\big) \\
&= \frac{\pi}{8}\Lambda^2(s + 2\,b + 2\,c) + \frac{\pi^2}{4}\Lambda^3(s\,b + s\,c + 2\,b\,c)\,.
\end{aligned}
\tag{2.9}
$$

Both $E_{D_{piston}}(\Lambda)$ and $E_{N_{piston}}(\Lambda)$ have no dependence on the plate separation $a$. This is due to a cancellation that has occurred between region I and II. The Casimir force on the piston has therefore no dependence on the cut-off $\Lambda$ (since the partial derivative with respect to $a$ of $E_{D_{piston}}(\Lambda)$ and $E_{N_{piston}}(\Lambda)$ is zero).

## 2.2   Cancellation in $d$ dimensions

In a $d$-dimensional Casimir piston, the piston has $d$–1 dimensions and divides again the volume into two regions: an interior region I and exterior region II. Without loss of generality, the direction in which the piston moves is chosen to be along the $L_1$ direction so that region I and II share the same lengths except for $L_1$. It is therefore convenient to write the Dirichlet and Neumann cut-off expressions (2.2) and (2.4) as a sum of two terms: one that includes $L_1$ and another which is independent of $L_1$ i.e.

$$
\begin{aligned}
E_D(\Lambda) &= \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m}\, m\, 2^m\, \pi^{\frac{m+1}{2}}\, \Gamma(\tfrac{m+1}{2})\, \Lambda^{m+1} \left( L_1\, \xi^d_{1,\,k_2,..,k_m} \prod_{i=2}^{m} L_{k_i} + \xi^d_{\substack{k_1,..,k_m \\ k_1 \neq 1}} \prod_{i=1}^{m} L_{k_i} \right) \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m}\, f(m) \left( L_1\, \xi^d_{1,\,k_2,..,k_m} \prod_{i=2}^{m} L_{k_i} + \xi^d_{\substack{k_1,..,k_m \\ k_1 \neq 1}} \prod_{i=1}^{m} L_{k_i} \right)
\end{aligned}
\tag{2.10}
$$

where $f(m) \equiv m \, 2^m \, \pi^{\frac{m+1}{2}} \, \Gamma(\frac{m+1}{2}) \, \Lambda^{m+1}$. The Neumann cut-off expression is given by

$$E_N(\Lambda) \equiv \frac{1}{2^{d+1}} \sum_{m=1}^{d} f(m) \left( L_1 \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right). \qquad (2.11)$$

Let the plate separation be $a$. In region I, $L_1 = a$ and in region II, $L_1 = s - a$ (the piston splits the length $s$ into $a$ and $s - a$ along the $L_1$ direction) (see fig. 1). To obtain the cut-off dependence for the $d$-dimensional Casimir piston we need to add the contributions from regions I and II:

$$\begin{aligned}
E_{D_{piston}}(\Lambda) &\equiv E_{D_1}(\Lambda) + E_{D_2}(\Lambda) \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m} f(m) \left( a \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right) \\
&+ \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m} f(m) \left( (s-a) \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right) \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} (-1)^{d+m} f(m) \left( s \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + 2 \, \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right)
\end{aligned} \qquad (2.12)$$

$$\begin{aligned}
E_{N_{piston}}(\Lambda) &\equiv E_{N_1}(\Lambda) + E_{N_2}(\Lambda) \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} f(m) \left( a \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right) \\
&+ \frac{1}{2^{d+1}} \sum_{m=1}^{d} f(m) \left( (s-a) \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right) \\
&= \frac{1}{2^{d+1}} \sum_{m=1}^{d} f(m) \left( s \, \xi_{1,\, k_2,..,k_m}^{d} \prod_{i=2}^{m} L_{k_i} + 2 \, \xi_{\substack{k_1,..,k_m \\ k_1 \neq 1}}^{d} \prod_{i=1}^{m} L_{k_i} \right)
\end{aligned} \qquad (2.13)$$

The cut-off expressions for the piston, $E_{D_{piston}}(\Lambda)$ and $E_{N_{piston}}(\Lambda)$, have no dependence on the plate separation $a$. Their derivatives with respect to $a$ is zero which implies that the Casimir force on the piston has no cut-off dependence in any dimension $d$. The hypersurface divergences have cancelled out in all dimensions in the Casimir piston scenario.

# 3   Casimir force formulas in the $d$-dimensional Casimir piston

Having proved the cancellation of hypersurface divergences in the $d$-dimensional Casimir piston, we now focus on the finite ($\Lambda$-independent) part of the Casimir energy. The finite part is

conveniently expressed as a sum of two terms: an analytical term composed of a finite sum over Riemann zeta and gamma functions and a remainder term $R_j$ composed of infinite sums over modified Bessel functions (though convergence is reached after summing a few terms). In appendix A we derive exact expressions for the finite part $E_{0_N}$ and $E_{0_D}$ of the Casimir energy in $d$ dimensions for Neumann and Dirichlet boundary conditions respectively. In this section, we state these expressions and use them to obtain the Neumann and Dirichlet Casimir force $F_N$ and $F_D$ for the $d$-dimensional Casimir piston. In appendix B we develop alternative expressions for the Casimir force (see discussion at the end of this section).

The finite part of the $d$-dimensional Casimir energy for Neumann and Dirichlet boundary conditions, $E_{0_N}$ and $E_{0_D}$, is given by (A.22) and (A.26) respectively:

$$E_{0_N} = -\frac{\pi}{2^{d+1}} \sum_{m=1}^{d} \sum_{j=0}^{m-1} 2^{d-m} \, \xi_{k_1,..,k_j}^{m-1} \frac{L_{k_1} \ldots L_{k_j}}{(L_m)^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2) + R_{N_j} \right) \qquad (3.14)$$

where the remainder $R_{N_j}$ is given by (A.23)

$$R_{N_j} = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i=-\infty \\ i=1,\ldots,j}}^{\infty} \frac{2 \, n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\left( 2\pi \, n \sqrt{(\ell_1 \frac{L_{k_1}}{L_m})^2 + \cdots + (\ell_j \frac{L_{k_j}}{L_m})^2} \, \right)}{\left[ (\ell_1 \frac{L_{k_1}}{L_m})^2 + \cdots + (\ell_j \frac{L_{k_j}}{L_m})^2 \right]^{\frac{j+1}{4}}} , \qquad (3.15)$$

and

$$E_{0_D} = \frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} (-1)^{d+j} \, \xi_{k_1,..,k_j}^{d-1} \frac{L_{k_1} \ldots L_{k_j}}{(L_d)^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2) + R_{D_j} \right) \qquad (3.16)$$

where $R_{D_j}$ is given by (A.27):

$$R_{D_j} = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i=-\infty \\ i=1,\ldots,j}}^{\infty} \frac{2 \, n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\left( 2\pi \, n \sqrt{(\ell_1 \frac{L_{k_1}}{L_d})^2 + \cdots + (\ell_j \frac{L_{k_j}}{L_d})^2} \, \right)}{\left[ (\ell_1 \frac{L_{k_1}}{L_d})^2 + \cdots + (\ell_j \frac{L_{k_j}}{L_d})^2 \right]^{\frac{j+1}{4}}} . \qquad (3.17)$$

The prime on the sum in (3.15) and (3.17) means that the case when all $\ell$'s are simultaneously zero ($\ell_1 = \ell_2 = \ldots = \ell_j = 0$) is to be excluded. There is an implicit summation over the $k_i$'s via the ordered symbol $\xi_{k_1,..,k_j}$ defined in (2.5). $R_{N_j}$ and $R_{D_j}$ do not depend only on $j$ but are also a function of the ratios of lengths, for example $R_{N_j} = R_{N_j}(L_{k_1}/L_m, \ldots, L_{k_j}/L_m)$. Therefore, the implicit summation over the $k_i$'s applies also to $R_{N_j}$ and $R_{D_j}$. For $j = 0$, $R_{N_j}$ and $R_{D_j}$ are defined to be zero and $\xi_{k_1,..,k_j}$ and $L_{k_j}$ are defined to be unity so that $\xi_{k_1,..,k_j}^{d-1} (L_{k_1} \ldots L_{k_j})/(L_d)^{j+1} = 1/L_d$ for $j = 0$.

To obtain the $d$-dimensional Casimir energy in the piston scenario we need to sum the contributions from region I and region II. In region I, let the length of the sides of the $d$-dimensional parallelepiped region be $a_1, a_2, \ldots, a_{d-1}, a$ where $a$ is the plate separation. In region I, we label the lengths $L_i$ such that $L_1 = a_1$, $L_2 = a_2$, etc. with $L_d = a$ ($L_d$ is equal to the plate separation). In region II, the length of the sides are the same as in region I except for the length $a$ which is replaced by the length $s - a$. The $d$ lengths are $s - a, a_1, a_2, \ldots, a_{d-1}$. For region II, we choose to label the lengths $L_i$ such that $L_1 = s - a$, $L_2 = a_1$, $L_3 = a_2, \ldots, L_d = a_{d-1}$. To calculate the Casimir force, we only need to keep terms in the Casimir energy that depend on the plate separation $a$: in region I, this means keeping terms with $L_d = a$ and in region II this means keeping terms with $L_1 = s - a$.

In region I, the $a$-dependent Casimir energy for Neumann boundary conditions is obtained by setting $m = d$ so that $L_m = L_d = a$ and setting $L_{k_j} = a_{k_j}$ in (3.14):

$$E_{0_{N_I}}(a) = -\frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} \xi_{k_1,..,k_j}^{d-1} \frac{a_{k_1} \ldots a_{k_j}}{a^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_{I_{N_j}} \right) \tag{3.18}$$

with $R_{I_{N_j}}$ given by

$$R_{I_{N_j}} = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i=-\infty \\ i=1,\ldots,j}}^{\infty} \frac{2\, n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\left( 2\pi\, n\, \sqrt{(\ell_1 \frac{a_{k_1}}{a})^2 + \cdots + (\ell_j \frac{a_{k_j}}{a})^2} \right)}{\left[ (\ell_1 \frac{a_{k_1}}{a})^2 + \cdots + (\ell_j \frac{a_{k_j}}{a})^2 \right]^{\frac{j+1}{4}}}. \tag{3.19}$$

A word on notation: the Roman numerals I and II will denote region I and II respectively while $j$ will be denoted via Arabic numerals $1, 2, 3$ etc. e.g. $R_{I_{N_1}}$ means the remainder(R) for Neumann(N) in region I with $j = 1$.

The finite part of the Casimir energy for Dirichlet boundary conditions in region I is obtained by setting $L_d = a$ and $L_{k_j} = a_{k_j}$ in (3.16):

$$E_{0_{D_I}} = \frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} (-1)^{d+j}\, \xi_{k_1,..,k_j}^{d-1} \frac{a_{k_1} \ldots a_{k_j}}{a^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_{I_{D_j}} \right) \tag{3.20}$$

where $R_{I_{D_j}}$ is given by (3.17):

$$R_{I_{D_j}} = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i=-\infty \\ i=1,\ldots,j}}^{\infty} \frac{2\, n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\left( 2\pi\, n\, \sqrt{(\ell_1 \frac{a_{k_1}}{a})^2 + \cdots + (\ell_j \frac{a_{k_j}}{a})^2} \right)}{\left[ (\ell_1 \frac{a_{k_1}}{a})^2 + \cdots + (\ell_j \frac{a_{k_j}}{a})^2 \right]^{\frac{j+1}{4}}}. \tag{3.21}$$

In region II, only terms with $L_1 = s - a$ contribute to the $a$-dependent Casimir energy. We therefore consider only the cases when $k_1$ is equal to 1 so that $L_{k_1} = s - a$. The rest of the

lengths ($j > 1$) are given by $L_{k_j} = a_{k_j-1}$ so that $L_2 = a_1, L_3 = a_2, \ldots, L_m = a_{m-1}$. We are interested in an exterior of infinite length so that the Casimir force in region II is calculated in the limit $s \to \infty$. For Neumann boundary conditions, the case ($j = 0, m = 1$) in (3.14) can be omitted because $L_m = L_1 = s - a$ appears in the denominator and yields a zero Casimir force in the limit of $s \to \infty$. The cases $j = 0$, $m > 1$ can also be dropped because they do not yield any terms with $L_1$ (with $j = 0$, the numerator is equal to unity and with $m > 1$, $L_m \neq L_1$). In region II, the lower limit of the sums in (3.14) therefore start at $j = 1$ and $m = 2$ yielding the following Neumann energy ($a$-dependent part) :

$$E_{0_{N_{II}}}(a) = -\frac{\pi}{2^{d+1}} \sum_{m=2}^{d} \sum_{j=1}^{m-1} 2^{d-m} \, \xi_{1,k_2,..,k_j}^{m-1} \, \frac{(s-a)\, a_{k_2-1} \ldots a_{k_j-1}}{(a_{m-1})^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2) + R_{II_{N_j}} \right)$$

(3.22)

with

$$R_{II_{N_j}} = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i = -\infty \\ i=1,\ldots,j}}^{\infty} \frac{2 \, n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\left( 2\pi \, n \sqrt{(\ell_1 \frac{s-a}{a_{m-1}})^2 + \cdots + (\ell_j \frac{a_{k_j-1}}{a_{m-1}})^2} \,\right)}{\left[ (\ell_1 \frac{s-a}{a_{m-1}})^2 + \cdots + (\ell_j \frac{a_{k_j-1}}{a_{m-1}})^2 \right]^{\frac{j+1}{4}}} .$$

(3.23)

For Dirichlet boundary conditions, the case $j = 0$ can be dropped from (3.16) because it does not yield any terms with $L_1$. The sum in (3.16) therefore starts at $j = 1$ and we set $L_{k_1} = s - a$ and $L_{k_j} = a_{k_j-1}$, with $L_d = a_{d-1}$:

$$E_{0_{D_{II}}}(a) = \frac{\pi}{2^{d+1}} \sum_{j=1}^{d-1} (-1)^{d+j} \, \xi_{1,..,k_j}^{d-1} \, \frac{(s-a) \ldots a_{k_j-1}}{(a_{d-1})^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2) + R_{II_{D_j}} \right) \quad (3.24)$$

with

$$R_{II_{D_j}} = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i = -\infty \\ i=1,\ldots,j}}^{\infty} \frac{2 \, n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\left( 2\pi \, n \sqrt{(\ell_1 \frac{s-a}{a_{d-1}})^2 + \cdots + (\ell_j \frac{a_{k_j-1}}{a_{d-1}})^2} \,\right)}{\left[ (\ell_1 \frac{s-a}{a_{d-1}})^2 + \cdots + (\ell_j \frac{a_{k_j-1}}{a_{d-1}})^2 \right]^{\frac{j+1}{4}}} .$$

(3.25)

It is now straightforward to calculate the Casimir forces in each region. The Casimir force contribution from region I for Neumann is

$$F_{N_I} = -\frac{\partial \, E_{0_{N_I}}(a)}{\partial \, a} = -\frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} \xi_{k_1,..,k_j}^{d-1} \, (a_{k_1} \ldots a_{k_j}) \, \frac{j+1}{a^{j+2}} \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2) - \frac{\partial R_{I_N}}{\partial \, a} \quad (3.26)$$

where $R_{I_N}$ is the remainder contribution given by

$$R_{I_N} = -\frac{\pi}{2^{d+1}} \sum_{j=1}^{d-1} \xi_{k_1,..,k_j}^{d-1} \, \frac{(a_{k_1} \ldots a_{k_j})}{a^{j+1}} \, R_{I_{N_j}} .$$

(3.27)

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The corresponding formula for Dirichlet are

$$F_{D_I} = -\frac{\partial E_{0_{D_I}}}{\partial a} = \frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} (-1)^{d+j} \, \xi_{k_1,..,k_j}^{d-1} \, (a_{k_1} \ldots a_{k_j}) \frac{j+1}{a^{j+2}} \, \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \zeta(j+2) - \frac{\partial R_{I_D}}{\partial a}$$

(3.28)

with $R_{I_D}$ given by

$$R_{I_D} = \frac{\pi}{2^{d+1}} \sum_{j=1}^{d-1} (-1)^{d+j} \, \xi_{k_1,..,k_j}^{d-1} \, \frac{(a_{k_1} \ldots a_{k_j})}{a^{j+1}} \, R_{I_{D_j}} \,.$$

(3.29)

The Casimir force from the exterior region II is obtained in the limit when $s$ tends to infinity. For the Neumann case one obtains

$$F_{N_{II}} = -\lim_{s\to\infty} \frac{\partial}{\partial a} E_{0_{N_{II}}}(a) = -\frac{\pi}{2^{d+1}} \sum_{m=2}^{d} \sum_{j=1}^{m-1} 2^{d-m} \, \xi_{1,k_2,..,k_j}^{m-1} \, \frac{a_{k_2-1} \ldots a_{k_j-1}}{(a_{m-1})^{j+1}} \, \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2)$$

$$- \frac{\pi}{2^{d+1}} \sum_{m=3}^{d} \sum_{j=2}^{m-1} 2^{d-m} \, \xi_{1,k_2,..,k_j}^{m-1} \, \frac{a_{k_2-1} \ldots a_{k_j-1}}{(a_{m-1})^{j+1}} \, R_{II_{N_j}}(\ell_1=0)$$

(3.30)

where we used the result[2] $-\lim_{s\to\infty} \frac{\partial}{\partial a}\big[(s-a)\,R_{II_{N_j}}\big] = R_{II_{N_j}}(\ell_1=0)$. $R_{II_{N_j}}(\ell_1=0)$ means $R_{II_{N_j}}$ evaluated with $\ell_1=0$. Note that the product $a_{k_2-1} \ldots a_{k_j-1}$ that appears in (3.30) is identically equal to one for $j=1$ so that $\xi_{1,k_2,..,k_j}^{m-1} \, (a_{k_2-1} \ldots a_{k_j-1})/(a_{m-1})^{j+1} = 1/(a_{m-1})^2$ for $j=1$.

For the Dirichlet case one obtains

$$F_{D_{II}} = -\lim_{s\to\infty} \frac{\partial}{\partial a} E_{0_{D_{II}}} = \frac{\pi}{2^{d+1}} \sum_{j=1}^{d-1} (-1)^{d+j} \, \xi_{1,k_2,..,k_j}^{d-1} \, \frac{a_{k_2-1} \ldots a_{k_j-1}}{(a_{d-1})^{j+1}} \, \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2)$$

$$+ \frac{\pi}{2^{d+1}} \sum_{j=2}^{d-1} (-1)^{d+j} \, \xi_{1,k_2,..,k_j}^{d-1} \, \frac{a_{k_2-1} \ldots a_{k_j-1}}{(a_{d-1})^{j+1}} \, R_{II_{D_j}}(\ell_1=0) \,.$$

(3.31)

---

[2] $-\lim_{s\to\infty} \frac{\partial}{\partial a}\big[(s-a)\,R_{II_{N_j}}\big] = \lim_{s\to\infty} R_{II_{N_j}} - \lim_{s\to\infty} (s-a)\frac{\partial}{\partial a} R_{II_{N_j}}$. The first term yields $R_{II_{N_j}}(\ell_1=0)$ since the modified Bessel functions decrease to zero exponentially in the limit $s\to\infty$ except when $\ell_1=0$ (as there is no $s$ dependence when $\ell_1=0$). The second term is zero because the derivative of the modified Bessel functions with respect to $a$ decrease exponentially to zero in the limit $s\to\infty$ when $\ell_1 \neq 0$. When $\ell_1=0$, $R_{II_{N_j}}$ no longer has any dependence on $a$ so that its derivative is zero identically.

The Casimir force $F_N$ and $F_D$ on the piston for Neumann and Dirichlet respectively is finally obtained by adding contributions from both region I and II:

$$F_N = F_{N_I} + F_{N_{II}} \quad ; \quad F_D = F_{D_I} + F_{D_{II}} . \tag{3.32}$$

Eq. (3.32) together with (3.26) and (3.30) for $F_{N_I}$ and $F_{N_{II}}$, and (3.28) and (3.31) for $F_{D_I}$ and $F_{D_{II}}$ respectively constitute our final result for the Casimir force on the piston in $d$ dimensions.

The modified Bessel functions $K_{\frac{j+1}{2}}$ that appear in $R_{I_{N_j}}$ or $R_{I_{D_j}}$ (Eqs. (3.19) and (3.21)) converge exponentially fast if the plate separation $a$ is the smallest of the $d$ lengths because the ratios $a_{k_i}/a$ that appear in the argument of the Bessel functions are then greater than or equal to 1. Only a few terms need to be summed to reach high accuracy and the result is also small in magnitude. This is why we can call $R$ a remainder. However, $R_{I_{N_j}}$ and $R_{I_{D_j}}$ can converge slowly and be large if $a$ is larger than the other lengths. In particular, the large $a$ limit when $a_{k_i}/a \ll 1$ would require a very large number of terms to be summed before convergence is reached. By making use of the invariance of the vacuum energy under permutation of lengths, we derive in appendix B alternative expressions that are more convenient to use when the plate separation $a$ is large. For Neumann and Dirichlet they are given by (B.10) and (B.13) respectively:

$$F_N^{alt} = -\frac{\pi}{24\,a^2} - \frac{\partial}{\partial a} R_{I_N}^{alt}(\ell_1 \neq 0) \tag{3.33}$$

and

$$F_D^{alt} = -\frac{\partial}{\partial a} R_{I_D}^{alt}(\ell_1 \neq 0) \tag{3.34}$$

where $R_{I_N}^{alt}(\ell_1 \neq 0)$ is given by (B.6) with (B.11) and $R_{I_D}^{alt}(\ell_1 \neq 0)$ is given by (B.8) with (B.14). The above compact formulas are applied in the next section in two and three dimensions where one can see explicitly how they are used.

The ratio of lengths that appear in the argument of the modified Bessel functions in (B.11) and (B.14) have $a$ in the numerator $(a/a_{k_i})$ in contrast to our original expressions (with ratio $a_{k_i}/a$). We have a useful $a \rightarrow 1/a$ duality: when $a$ is large a long computation with the original expressions can be trivial using the alternative expressions and vice versa when $a$ is small. The invariance of the vacuum energy under permutations of the $d$ lengths was used to derive the alternative expressions and the duality can be traced to this symmetry. Note that regardless of the size of the plate separation $a$, we would want to label the other $d-1$ lengths such that $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_{d-1}$ to reach the quickest convergence.

The Casimir force on the piston is negative (attractive) in all dimensions for both Neumann and Dirichlet boundary conditions and ranges from $-\infty$ (in the limit $a \rightarrow 0$) to 0 (in the limit $a \rightarrow \infty$). The limit as $a \rightarrow 0$ is easily determined using the original expressions (3.32). In the limit $a \rightarrow 0$, $F_{N_I}$ and $F_{D_I}$ given by (3.26) and (3.28) tend to $-1/a^{d+1}$ ( $\partial R_{I_N}/\partial a$ and $\partial R_{I_D}/\partial a$ tend to zero). $F_{N_{II}}$ and $F_{D_{II}}$ have no dependence on $a$ so that $F_N = F_{N_I} + F_{N_{II}}$

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and $F_D = F_{D_I} + F_{D_{II}}$ tend towards $-1/a^{d+1}$ and hence $-\infty$ as $a \to 0$. To determine the limit as $a \to \infty$, it is easiest to use the alternative expressions. As already discussed at the end of appendix B, $F_N^{alt}$ and $F_D^{alt}$ given by (3.33) and (3.34) tend to zero in that limit because the modified Bessel functions that appear in $R_{I_N}^{alt}(\ell_1 \neq 0)$ (Eqs. (B.6) and (B.11)) and $R_{I_D}^{alt}(\ell_1 \neq 0)$ (Eqs. (B.8) and (B.14)) decrease exponentially fast to zero as $a \to \infty$ (since $\ell_1 \neq 0$, when $a \to \infty$ the argument of the Bessel functions tend to infinity). The rapid decrease to zero can be seen in the two and three-dimensional plots of the next section (fig. 2 and fig. 3).

# 4 Application: two and three-dimensional Casimir piston for Neumann boundary conditions

Exact results for massless scalar fields in the two and three-dimensional Casimir piston for Dirichlet boundary conditions were first obtained in Refs. [5, 8] and recently exact results for 3D Neumann (as well as Dirichlet) were obtained in [12]. We apply our $d$-dimensional formulas (both expressions) to the two and three-dimensional Casimir piston with Neumann boundary conditions. The 2D Neumann results are new and fill a gap in the literature. For 3D Neumann, our first expression looks similar in form to the one recently derived in [12] and is in numerical agreement with it providing an independent confirmation of our results. Our alternative 3D Neumann expression looks quite different in form from the first expression and yields the same numerical results but is more useful (converges more quickly) at large plate separation $a$.

## 4.1 Two dimensions

In $d$ dimensions the lengths of the parallelepiped are $a, a_1, a_2, \ldots, a_{d-1}$ with $a$ being the plate separation. In two dimensions the lengths are then $a$ and $a_1$ (we set $a_1 = b$ so that the geometry is an $a \times b$ rectangle). The Casimir force contribution from region I is obtained by setting $d = 2$ in (3.26):

$$
\begin{aligned}
F_{N_I} &= -\frac{\pi}{8} \sum_{j=0}^{1} \xi_{k_1,..,k_j}^{1} \left(a_{k_1} \ldots a_{k_j}\right) \frac{j+1}{a^{j+2}} \, \Gamma(\tfrac{j+2}{2}) \, \pi^{\frac{-j-4}{2}} \, \zeta(j+2) - \frac{\partial R_{I_N}}{\partial a} \\
&= -\frac{\pi}{48 \, a^2} - \frac{\zeta(3) \, b}{8 \, \pi \, a^3} + \frac{1}{2} \frac{\partial}{\partial a} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{n}{\ell} \frac{1}{a} K_1\big(2 \, \pi \, n \, \ell \, b/a\big)
\end{aligned}
\tag{4.35}
$$

where $R_{I_N}$ is obtained from (3.27):

$$
R_{I_N} = -\frac{\pi}{8} \frac{b}{a^2} \, R_{I_{N_1}}(b/a) = -\frac{1}{2} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{n}{\ell} \frac{1}{a} K_1\big(2 \, \pi \, n \, \ell \, b/a\big) .
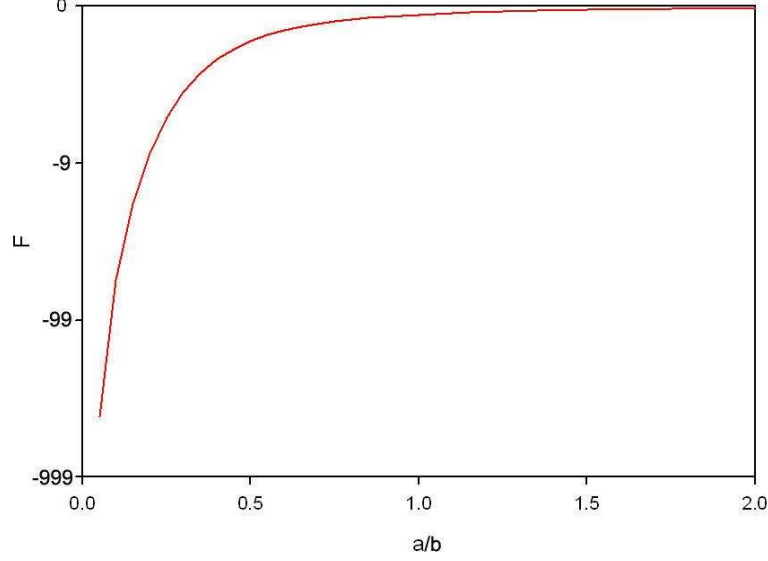\tag{4.36}
$$

Figure 2: Neumann Casimir force $F$ versus $a/b$ where $a$ is the plate separation and $b$ the second side of a two-dimensional rectangular region. The force is in units of $1/b^2$. The force is negative with magnitude decreasing quickly to zero as $a/b$ increases.

$R_{I_{N_1}}(b/a)$ is obtained from (3.19) and means $R_{I_{N_1}}$ is a function of $b/a$.

The Casimir force contribution from region II is obtained by setting $d = 2$ in (3.30). In the first double sum, there is only the term $j = 1, m = 2$ to consider. The second double sum is zero (it is nonzero only starting at $d = 3$). We obtain the simple expression

$$F_{N_{II}} = -\frac{\zeta(3)}{16\,\pi\,b^2}\,. \tag{4.37}$$

By summing $F_{N_I}$ and $F_{N_{II}}$ we obtain the Casimir force $F_N$ on the piston:

$$F_N = -\frac{\pi}{48\,a^2} - \frac{\zeta(3)\,b}{8\,\pi\,a^3} - \frac{\zeta(3)}{16\,\pi\,b^2} + \frac{1}{2}\frac{\partial}{\partial a}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{n}{\ell}\,\frac{1}{a}\,K_1\big(2\,\pi\,n\,\ell\,b/a\big)\,. \tag{4.38}$$

As an example, the above expression yields $F_N = -0.1342935575/b^2$ for the case of a square $(a = b)$.

An alternative expression $F_N^{alt}$ for the Casimir force can be obtained via (3.33). For $d = 2$ we

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obtain

$$
\begin{aligned}
F_N^{alt} &= -\frac{\pi}{24\,a^2} - \frac{\partial}{\partial a} R_{I_N}^{alt}(\ell_1 \neq 0) \\
&= -\frac{\pi}{24\,a^2} + \frac{1}{2\,b} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{n}{\ell} K_1'\big( 2\pi\, n\, \ell\, a/b\big)
\end{aligned}
\tag{4.39}
$$

where the prime on the modified Bessel function means partial derivative with respect to $a$ i.e. $K'(x) \equiv \frac{\partial}{\partial a} K(x)$. $R_{I_N}^{alt}(\ell_1 \neq 0)$ is obtained from the $j{=}1$, $m{=}2$ term in (B.6):

$$
R_{I_N}^{alt}(\ell_1 \neq 0) = -\frac{\pi\,a}{8\,b^2} R_{I_{N_1}}^{alt}(\ell_1 \neq 0) = -\frac{1}{2\,b} \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \frac{n}{\ell} K_1\big( 2\pi\, n\, \ell\, a/b\big) .
\tag{4.40}
$$

$R_{I_{N_1}}^{alt}(\ell_1{\neq}0)$ is obtained from the $j{=}1$, $m{=}2$ term in (B.11).

The two expressions, (4.38) and (4.39), yield the same value for the Casimir force on the piston and are valid for any values of $a$ and $b$. However, computationally, expression (4.38) is better to use when $a$ is small (i.e. $a/b{<}1$), whereas (4.39) is better to use when $a$ is large ($b/a{<}1$). This is the simplest case of the $a \to 1/a$ duality that was discussed last section.

The Casimir force on the piston is negative (attractive) and ranges from $-\infty$ (in the limit $a{\to}0$) to $0$ (in the limit $a{\to}\infty$). A plot of $F$ versus $a/b$ (in units of $1/b^2$) is shown in fig. 2.

## 4.2  Three dimensions

In $d$-dimensions we have the $d$ lengths $a, a_1, a_2, \ldots, a_{d-1}$ with $a$ the plate separation. In three dimensions the three lengths are then $a, a_1$ and $a_2$. For the three-dimensional Casimir piston it has become customary to use $a, b$ and $c$ for the lengths and we therefore set $a_2 = b$ and $a_1 = c$. The Casimir force contribution from region I is obtained by setting $d = 3$ in (3.26)

$$
\begin{aligned}
F_{N_I} &= -\frac{\pi}{16} \sum_{j=0}^{2} \xi_{k_1,..,k_j}^2 \, (a_{k_1} \ldots a_{k_j}) \frac{j+1}{a^{j+2}} \Gamma(\tfrac{j+2}{2})\, \pi^{\frac{-j-4}{2}} \zeta(j+2) - R_{I_N}' \\
&= -\frac{\pi^2\,b\,c}{480\,a^4} - \frac{\zeta(3)\,(b+c)}{16\,\pi\,a^3} - \frac{\pi}{96\,a^2} - R_{I_N}'
\end{aligned}
\tag{4.41}
$$

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where $R_{I_N}$ is given by (3.27) and (3.19):

$$R_{I_N} = -\frac{\pi}{16}\left[\frac{c}{a^2}R_{I_{N_1}}(c/a) + \frac{b}{a^2}R_{I_{N_1}}(b/a) + \frac{2c}{b^2}R_{I_{N_1}}(c/b) + \frac{bc}{a^3}R_{I_{N_2}}(c/a,b/a)\right]$$

$$= -\frac{1}{4}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{n}{\ell}\left[\frac{1}{a}K_1\big(2\pi n\ell c/a\big) + \frac{1}{a}K_1\big(2\pi n\ell b/a\big) + \frac{2}{b}K_1\big(2\pi n\ell c/b\big)\right]$$

$$- \frac{bc}{8a^3}\sum_{n=1}^{\infty}{\sum_{\ell_1,\ell_2=-\infty}^{\infty}}'\frac{n^{3/2}K_{3/2}\Big(2\pi n\sqrt{\big(\frac{\ell_1 c}{a}\big)^2 + \big(\frac{\ell_2 b}{a}\big)^2}\Big)}{\left[\big(\frac{\ell_1 c}{a}\big)^2 + \big(\frac{\ell_2 b}{a}\big)^2\right]^{3/4}}\,.$$

(4.42)

$R_{I_{N_1}}(c/a)$ means $R_{I_{N_1}}$ is a function of $c/a$ and the prime above the sum means that the case $\ell_1 = \ell_2 = 0$ is to be excluded from the sum.

The Casimir force contribution from region II is obtained by setting $d = 3$ in (3.30):

$$F_{N_{II}} = -\frac{\pi}{16}\sum_{m=2}^{3}\sum_{j=1}^{m-1}2^{3-m}\,\xi_{1,k_2,..,k_j}^{m-1}\frac{a_{k_2-1}\cdots a_{k_j-1}}{(a_{m-1})^{j+1}}\Gamma(\tfrac{j+2}{2})\,\pi^{\frac{-j-4}{2}}\zeta(j+2) - \frac{\pi}{16}\frac{c}{b^3}R_{II_{N_2}}(\ell_1=0)$$

$$= -\frac{\pi^2 c}{1440\,b^3} - \frac{\zeta(3)}{16\,\pi}\Big(\frac{1}{c^2} + \frac{1}{2\,b^2}\Big) - \frac{c}{4\,b^3}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\Big(\frac{n\,b}{\ell\,c}\Big)^{3/2}K_{3/2}(2\,\pi\,n\,\ell\,c/b)$$

(4.43)

where (3.23) with $j=2$ and $m=3$ was used to obtain

$$R_{II_{N_2}}(\ell_1=0) = \frac{4}{\pi}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\Big(\frac{n\,b}{\ell\,c}\Big)^{3/2}K_{3/2}(2\,\pi\,n\,\ell\,c/b)\,.$$

(4.44)

The Casimir force $F_N$ on the piston for Neumann boundary conditions in three dimensions is obtained by summing $F_{N_I}$ and $F_{N_{II}}$:

$$F_N = -\frac{\pi^2\,b\,c}{480\,a^4} - \frac{\zeta(3)\,(b+c)}{16\,\pi\,a^3} - \frac{\pi}{96\,a^2} - R'_{I_N}$$

$$- \frac{\pi^2 c}{1440\,b^3} - \frac{\zeta(3)}{16\,\pi}\Big(\frac{1}{c^2} + \frac{1}{2\,b^2}\Big) - \frac{c}{4\,b^3}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\Big(\frac{n\,b}{\ell\,c}\Big)^{3/2}K_{3/2}(2\,\pi\,n\,\ell\,c/b)$$

(4.45)

where $R_{I_N}$ is given by (4.42) and $R'_{I_N} \equiv \partial R_{I_N}/\partial a$. Note that the third term in square brackets in (4.42) depends only on $b$ and $c$ and can be dropped when evaluating $R'_{I_N}$. For the case of a cube ($a = b = c$), Eq. (4.45) yields $F_N = -0.1380999/c^2$. As in two dimensions, the force $F_N$
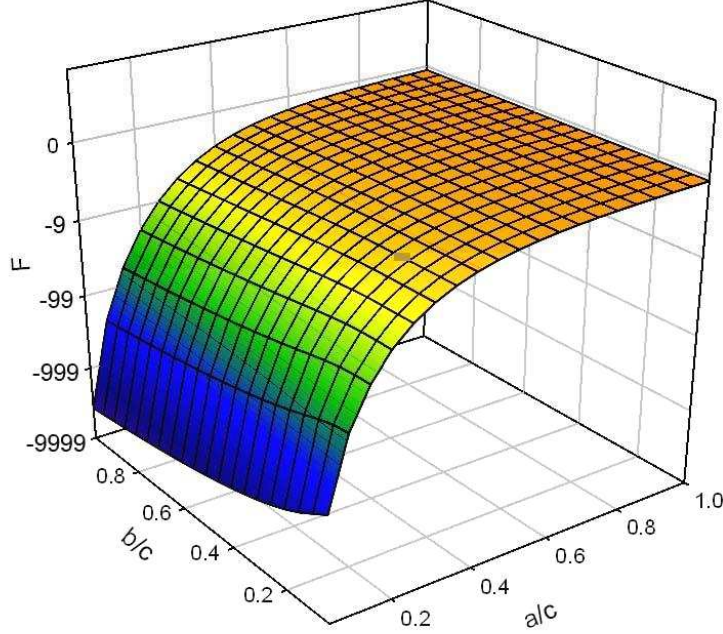
Figure 3: 3D plot of Neumann Casimir force $F_N$ versus $a/c$ and $b/c$. The force is in units of $1/c^2$. The force is large and negative at small values of $a/c$ and remains negative with its magnitude decreasing quickly to zero as $a/c$ increases. The value of $b/c$ shifts the magnitude of the force towards larger values as it increases.

is negative and ranges from $-\infty$ (in the limit $a \to 0$) to 0 (in the limit $a \to \infty$). A 3D plot of $F_N$ versus $a/c$ and $b/c$ (in units of $1/c^2$) is shown in fig. 3.

Our exact expression (4.45) looks similar in form to the one derived in [12] and is in numerical agreement with it. Moreover, in the small $a$ limit, $R'_{I_N}$ is exponentially suppressed (exactly zero in the limit $a \to 0$) and when $b = c$, the second row in (4.45) yields $0.0429965/c^2$ in agreement with the Neumann results in both [7] and [12]. This provides an independent confirmation of our results.

An alternative expression $F_N^{alt}$ for the Casimir force can be readily obtained by substituting

$d = 3$ in (3.33) and using (B.6) and (B.11):

$$
\begin{aligned}
F_N^{alt} &= -\frac{\pi}{24\,a^2} - \frac{\partial}{\partial a} R_{I_N}^{alt}(\ell_1 \neq 0) \\
&= -\frac{\pi}{24\,a^2} + \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{n}{4\,\ell} \left( \frac{2}{c} K_1'(2\,\pi\,n\,\ell\,a/c) + \frac{1}{b} K_1'(2\,\pi\,n\,\ell\,a/b) \right) \\
&\quad + \frac{\partial}{\partial a} \left[ \frac{a\,c}{4\,b^3} \sum_{n=1}^{\infty} \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=-\infty}^{\infty} n^{3/2} \frac{K_{3/2}\left(2\,\pi\,n\,\sqrt{(\ell_1\,a/b)^2 + (\ell_2\,c/b)^2}\right)}{\left[(\ell_1\,a/b)^2 + (\ell_2\,c/b)^2\right]^{3/4}} \right].
\end{aligned}
\tag{4.46}
$$

The prime above $K$ denotes partial derivative with respect to $a$. Note that the sum over $\ell_2$ includes $\ell_2 = 0$ (since the sum over $\ell_2$ contains no prime above it). Equation (4.46) is our alternative expression for the exact Casimir force on the piston in three dimensions for Neumann boundary conditions. It is valid for any values of the lengths $a, b$ and $c$ and yields the same Casimir force as the original expression (4.45). However, in the argument of the modified Bessel functions the plate separation $a$ now appears in the numerator making the alternative expression converge quickly for large $a$. A long computation with the original expression when $a$ is large can converge exponentially fast with the alternative expression and vice versa when $a$ is small. This is a nontrivial case of the $a \to 1/a$ duality already encountered in two dimensions and discussed in general in the last section. Note that we are free to label the base such that $c \geq b$ to obtain the best convergence.

# 5 Summary and discussion

By applying a multidimensional cut-off technique we obtained expressions for the cut-off dependent part of the Casimir energy for parallelepiped geometries in any spatial dimension $d$ and showed explicitly that nonrenormalizable hypersurface divergences cancel in the Casimir piston scenario in all dimensions. We then obtained exact expressions for the $d$-dimensional Casimir force on a piston for the case of massless scalar fields obeying Dirichlet and Neumann boundary conditions. As an example, we applied the $d$-dimensional formulas to the 2D and 3D piston with Neumann boundary conditions. The two main features of the Casimir piston originally mentioned by Cavalcanti [5] for a 2D rectangular geometry, namely the cancellation of the surface divergences and the negative Casimir force on the piston, were shown to hold true in all dimensions $d$. We obtained two different expressions for the $d$-dimensional Casimir force. The Casimir energy is clearly invariant under permutations of the $d$ lengths of the parallelepiped. This symmetry is trivial but its application is very useful: one can derive alternative expressions for the Casimir force that converge quickly compared to the original expressions when the plate separation $a$ is large.

It would be interesting to generalize our $d$-dimensional results to include arbitrary cross sections and thermal corrections. For scalar fields in three dimensions this has been recently considered in [12] via the optical path technique. The scalar field results were then used to obtain the electromagnetic (EM) Casimir energies with perfect metallic boundary conditions [12]. It would be worthwhile to see how the 3D alternative expressions for massless scalar fields derived here for Neumann and for Dirichlet elsewhere [8] can be modified to include thermal corrections. These could then be used to obtain 3D alternative expressions for the thermal corrections to the EM case.

## A  Cut-off dependent and finite parts of the regularized vacuum energy: periodic, Dirichlet and Neumann boundary conditions

We consider a massless scalar field confined to a $d$-dimensional parallelepiped region with arbitrary lengths $L_1, ..., L_d$ obeying periodic, Neumann and Dirichlet boundary conditions. Our goal is to include the cut-off dependent and finite parts of the vacuum energy regularized via a multidimensional cut-off technique [6]. This appendix naturally divides into two parts. We first consider periodic boundary conditions and make use of formulas found in section 2 and appendix B of [6]. In particular, we determine explicitly the $d$-dimensional cut-off dependence in the expression for the regularized vacuum energy. In contrast to dimensional or zeta-function regularization, the multidimensional cut-off technique performs no renormalization. The second part consists of finding the regularized vacuum energy for the Dirichlet and Neumann cases. This is obtained by summing over the vacuum energy of the periodic case.

The vacuum energy for periodic boundary conditions regularized using a cut-off $\lambda$ is given by [6]

$$
\tilde{E}_p\left(d,\lambda\right) = \pi \sum_{\substack{n_i=-\infty \\ i=1,...,d}}^{\infty} \sqrt{\frac{n_1^2}{L_1^2} + \cdots + \frac{n_d^2}{L_d^2}}\, \mathrm{e}^{-\lambda \sqrt{\frac{n_1^2}{L_1^2}+\cdots+\frac{n_d^2}{L_d^2}}} = -\pi\, \partial_\lambda \sum_{\substack{n_i=-\infty \\ i=1,...,d}}^{\infty} \mathrm{e}^{-\lambda \sqrt{\frac{n_1^2}{L_1^2}+\cdots+\frac{n_d^2}{L_d^2}}}
$$

$$
= -\pi\, \partial_\lambda \left( 1 + \sideset{}{'}\sum_{n_1=-\infty}^{\infty} \mathrm{e}^{-\lambda \sqrt{\frac{n_1^2}{L_1^2}}} + \sideset{}{'}\sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \mathrm{e}^{-\lambda \sqrt{\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}}} + \cdots + \sideset{}{'}\sum_{n_d=-\infty}^{\infty} \sum_{\substack{n_i=-\infty \\ i=1,...,d-1}}^{\infty} \mathrm{e}^{-\lambda \sqrt{\frac{n_1^2}{L_1^2}+\cdots+\frac{n_d^2}{L_d^2}}} \right)
$$

$$
= -\pi \sum_{j=0}^{d-1} \partial_\lambda \Lambda_j(\lambda)
$$

(A.1)

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where

$$\Lambda_j(\lambda) \equiv \sideset{}{'}\sum_{n=-\infty}^{\infty} \sum_{\substack{n_i=-\infty \\ i=1,\dots,j}}^{\infty} e^{-\lambda \sqrt{\frac{n^2}{L_{j+1}^2} + \frac{n_1^2}{L_1^2} + \cdots + \frac{n_j^2}{L_j^2}}}. \tag{A.2}$$

The prime on the sum over $n$ means that $n = 0$ is excluded from the sum. The function $\partial_\lambda \Lambda_j(\lambda)$ can be expressed in the following form [6] (in the limit $\lambda \to 0$)

$$\partial_\lambda \Lambda_j(\lambda) = \frac{L_1 \dots L_j}{(L_{j+1})^{j+1}} \left( 2^{j+1} \partial_{\lambda'} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\lambda' \sqrt{n^2 + x_1^2 + \cdots + x_j^2}} \, dx_1 \dots dx_j + R_j \right) \tag{A.3}$$

where $\lambda' \equiv \lambda/L_{j+1}$ and $L_1 \dots L_j$ is a product of lengths i.e. $\prod_{i=1}^{j} L_i$. This product is defined to be unity for the special case of $j = 0$. $R_j$ is given by [6]

$$R_j = \sum_{n=1}^{\infty} \sideset{}{'}\sum_{\substack{\ell_i=-\infty \\ i=1,\dots,j}}^{\infty} \frac{2 \left(n L_{j+1}\right)^{\frac{j+1}{2}}}{\pi \left[(\ell_1 L_1)^2 + \cdots + (\ell_j L_j)^2\right]^{\frac{j+1}{4}}} K_{\frac{j+1}{2}} \left( \frac{2\pi n}{L_{j+1}} \sqrt{(\ell_1 L_1)^2 + \cdots + (\ell_j L_j)^2} \right). \tag{A.4}$$

$R_j$ starts at $j = 1$ (it is zero for $j = 0$). The functions $K_{(j+1)/2}$ are modified Bessel functions and the prime on the sum means that the case where all the $\ell_i$'s are zero is excluded. Via the Euler-Maclaurin formula, the integral term in the round brackets in (A.3) can be decomposed into a cut-off dependent term (which diverges as $\lambda \to 0$) and a finite term which is independent of the cut-off (see section 2 of [6]):

$$2^{j+1} \partial_{\lambda'} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\lambda' \sqrt{n^2 + x_1^2 + \cdots + x_j^2}} \, dx_1 \dots dx_j$$

$$= \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) \; + \; \frac{j}{\lambda'^{j+1}} 2^j \pi^{\frac{j-1}{2}} \Gamma(\tfrac{j+1}{2}) - \frac{j+1}{\lambda'^{j+2}} 2^{j+1} \pi^{\frac{j}{2}} \Gamma(\tfrac{j+2}{2})$$

$$= \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) \; + \; \frac{j}{\lambda^{j+1}} 2^j \pi^{\frac{j-1}{2}} \Gamma(\tfrac{j+1}{2}) (L_{j+1})^{j+1} - \frac{(j+1)}{\lambda^{j+2}} 2^{j+1} \pi^{\frac{j}{2}} \Gamma(\tfrac{j+2}{2}) (L_{j+1})^{j+2}. \tag{A.5}$$

Substituting (A.5) into (A.3), the regularized vacuum energy (A.1) for periodic boundary

conditions is given by ( as $\lambda \to 0$)

$$
\tilde{E}_{p_{L_1 \dots L_d}}(d,\lambda) = -\pi \sum_{j=0}^{d-1} \frac{L_1 \dots L_j}{(L_{j+1})^{j+1}} \left[ \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j \right.
$$

$$
\left. + \frac{j}{\lambda^{j+1}} 2^j \pi^{\frac{j-1}{2}} \Gamma(\tfrac{j+1}{2})(L_{j+1})^{j+1} - \frac{(j+1)}{\lambda^{j+2}} 2^{j+1} \pi^{\frac{j}{2}} \Gamma(\tfrac{j+2}{2})(L_{j+1})^{j+2} \right]
$$

$$
= -\pi \sum_{j=0}^{d-1} \frac{L_1 \dots L_j}{(L_{j+1})^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j \right) + (L_1 \dots L_d)\frac{d}{\lambda^{d+1}} 2^d \pi^{\frac{d+1}{2}} \Gamma(\tfrac{d+1}{2})
$$

$$
= \frac{-\pi}{6\,L_1} - \frac{\zeta(3)}{2\pi} \frac{L_1}{L_2^2} - \frac{\pi^2}{90} \frac{L_1 L_2}{L_3^3} + \dots - R_1 \frac{\pi L_1}{L_2^2} - R_2 \frac{\pi L_1 L_2}{L_3^3} + \dots + \frac{d}{\lambda^{d+1}} 2^d \pi^{\frac{d+1}{2}} \Gamma(\tfrac{d+1}{2})(L_1 \dots L_d)
$$
(A.6)

The notation $\tilde{E}_{p_{L_1,\dots,L_d}}(d,\lambda)$ is a compact way of stating that the regularized vacuum energy $\tilde{E}_p$ is a function of the dimension $d$, the cut-off parameter $\lambda$ and the lengths $L_1, \dots, L_d$.

The regularized vacuum energy for Dirichlet and Neumann boundary conditions can be expressed as a sum over the periodic energies $\tilde{E}_p$ [8, 24]:

$$
\tilde{E}_{\binom{N}{D}} = \frac{1}{2^{d+1}} \sum_{m=1}^{d} (\pm 1)^{d+m} \xi^d_{k_1,..,k_m} \tilde{E}_{p_{L_{k_1},\dots,L_{k_m}}}(m,\lambda)
$$
(A.7)

where the plus and negative signs correspond to the Neumann $(N)$ and Dirichlet $(D)$ cases respectively. $\xi^d_{k_1,..,k_m}$ is called the ordered symbol and is defined by [20]

$$
\xi^d_{k_1,..,k_m} = \begin{cases} 1 & \text{if } k_1 < k_2 < \dots < k_m \,;\, 1 \le k_m \le d \\ 0 & \text{otherwise .} \end{cases}
$$
(A.8)

The $k_i$'s are positive integers that can run from 1 to a maximum value of $d$. The ordered symbol $\xi^d_{k_1,..,k_m}$ ensures that the *implicit summation* over the $k_i$'s is over all distinct sets $\{k_1, .., k_m\}$ under the constraint that $k_1 < k_2 < \dots < k_m$. The superscript $d$ specifies the maximum value of $k_m$. $E_{p_{L_{k_1},..,L_{k_m}}}(\lambda,m)$ is obtained from (A.6) by replacing $d$ by $m$ and $L_1$ by $L_{k_1}$, $L_2$ by $L_{k_2}$, etc. When substituted into (A.7) one obtains

$$
E_{\binom{N}{D}} = \frac{-\pi}{2^{d+1}} \sum_{m=1}^{d} (\pm 1)^{d+m} \xi^d_{k_1,..,k_m} \sum_{j=0}^{m-1} \frac{L_{k_1} \dots L_{k_j}}{(L_{k_{j+1}})^{j+1}} \left( \Gamma(\tfrac{j+2}{2}) \pi^{\frac{-j-4}{2}} \zeta(j+2) + R_j \right)
$$

$$
+ \frac{1}{2^{d+1}} \sum_{m=1}^{d} (\pm 1)^{d+m} \xi^d_{k_1,..,k_m} (L_{k_1} \dots L_{k_m}) \frac{m}{\lambda^{m+1}} 2^m \pi^{\frac{m+1}{2}} \Gamma(\tfrac{m+1}{2})
$$
(A.9)

where $R_j$ is now the function (A.4) with $L_1$ replaced by $L_{k_1}$, $L_2$ by $L_{k_2}$, etc. It is convenient to define the function

$$f_{j\ k_1,..,k_{j+1}} \equiv \frac{L_{k_1}\ldots L_{k_j}}{(L_{k_{j+1}})^{j+1}}\left(\Gamma(\tfrac{j+2}{2})\,\pi^{\frac{-j-4}{2}}\,\zeta(j+2)+R_j\right). \qquad (A.10)$$

Using (A.10) and rearranging the limits on $m$ and $j$, we can express (A.9) in the following compact form (we write out separately the Dirichlet and Neumann cases)

$$\tilde{E}_D = \frac{-\pi}{2^{d+1}}\sum_{j=0}^{d-1}\sum_{m=j+1}^{d}(-1)^{d+m}\,\xi^d_{k_1,..,k_m}\,f_{j\ k_1,...,k_{j+1}} + E_D(\Lambda) \qquad (A.11)$$

$$\tilde{E}_N = \frac{-\pi}{2^{d+1}}\sum_{j=0}^{d-1}\sum_{m=j+1}^{d}\xi^d_{k_1,..,k_m}\,f_{j\ k_1,...,k_{j+1}} + E_N(\Lambda) \qquad (A.12)$$

where the functions $E_D(\Lambda)$ and $E_N(\Lambda)$ are the cut-off dependent terms for the Dirichlet and Neumann cases respectively obtained from the second row in (A.9) (we now work with the cut-off $\Lambda \equiv 1/\lambda$ instead of $\lambda$ so that the divergent limit $\lambda \to 0$ is replaced by $\Lambda \to \infty$ which is the more customary notation):

$$E_D(\Lambda) \equiv \frac{1}{2^{d+1}}\sum_{m=1}^{d}(-1)^{d+m}\,\xi^d_{k_1,..,k_m}\,(L_{k_1}\ldots L_{k_m})\,m\,2^m\,\pi^{\frac{m+1}{2}}\,\Gamma(\tfrac{m+1}{2})\,\Lambda^{m+1} \quad (A.13)$$

$$E_N(\Lambda) \equiv \frac{1}{2^{d+1}}\sum_{m=1}^{d}\xi^d_{k_1,..,k_m}\,(L_{k_1}\ldots L_{k_m})\,m\,2^m\,\pi^{\frac{m+1}{2}}\,\Gamma(\tfrac{m+1}{2})\,\Lambda^{m+1} \qquad (A.14)$$

Note that in (A.12) the limits on $m$ and $j$ in the double sum have been rearranged compared to (A.9). We can decompose $\xi^d_{k_1,..,k_m}$ into a sum of two terms: $\xi^{d-1}_{k_1,..,k_{m-1},d}+\xi^{d-1}_{k_1,..,k_m}$. In the first term, $k_m$ is set to its maximum value of $d$ and the implicit sum is over the remaining $k_i$'s with the maximum value of $k_{m-1}$ equal to $d-1$ (hence the superscript $d-1$). The second term contains the remaining implicit summation with the maximum value of $k_m$ equal to $d-1$ (hence there is a superscript $d-1$ in the second term as well). For the case $m = d$, the decomposition yields only one term $\xi^d_{k_1,..,k_d} = \xi^{d-1}_{k_1,..,k_{d-1},d} + 0$ since $k_d$ can only be equal to $d$.

With this decomposition the sum over $m$ becomes

$$\sum_{m=j+1}^{d}(\pm 1)^{d+m}\,\xi^d_{k_1,..,k_m} = \sum_{m=j+1}^{d}(\pm 1)^{d+m}\left[\xi^{d-1}_{k_1,..,k_{m-1},d}+\xi^{d-1}_{k_1,..,k_m}\right]. \qquad (A.15)$$

There are two separate cases to evaluate above: the plus sign (the Neumann case) and the minus sign (the Dirichlet case). The Dirichlet case has already been calculated in [20] and the

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result is

$$\sum_{m=j+1}^{d} (-1)^{d+m}\, \xi^{d}_{k_1,..,k_m} = (-1)^{d+j+1}\, \xi^{d-1}_{k_1,..,k_j,d}. \tag{A.16}$$

For the Neumann case one obtains

$$\sum_{m=j+1}^{d} \xi^{d}_{k_1,..,k_m} = \xi^{d-1}_{k_1,..,k_j,d} + (\,\xi^{d-1}_{k_1,..,k_{j+1}} + \xi^{d-1}_{k_1,..,k_{j+1},d}\,)$$
$$+ (\,\xi^{d-1}_{k_1,..,k_{j+2}} + \xi^{d-1}_{k_1,..,k_{j+2},d}\,) + \ldots + (\,\xi^{d-1}_{k_1,..,k_{d-1}} + \xi^{d-1}_{k_1,..,k_{d-1},d}\,). \tag{A.17}$$

Each pair of round brackets contains the sum of two terms which are equal. For example, consider the first pair of round brackets $(\xi^{d-1}_{k_1,..,k_{j+1}} + \xi^{d-1}_{k_1,..,k_{j+1},d})$. The fact that $k_{j+2}$ is equal to $d$ in the second term is irrelevant since the summation over $f_{j\,k_1,..,k_{j+1}}$ in (A.12) ends at $k_{j+1}$ for a given $j$. Therefore, $\xi^{d-1}_{k_1,..,k_{j+1},d}$ is equal to $\xi^{d-1}_{k_1,..,k_{j+1}}$. The same logic applies to the other pairs of round brackets. Equation (A.17) reduces to a recursion relation

$$\sum_{m=j+1}^{d} \xi^{d}_{k_1,..,k_m} = \xi^{d-1}_{k_1,..,k_j,d} + 2 \sum_{m=j+1}^{d-1} \xi^{d-1}_{k_1,..,k_m}$$
$$= \xi^{d-1}_{k_1,..,k_j,d} + 2 \left(\xi^{d-2}_{k_1,..,k_j,d-1} + 2 \sum_{m=j+1}^{d-2} \xi^{d-2}_{k_1,..,k_m}\right). \tag{A.18}$$

Applying the above recursion repeatedly (another $d-j-2$ times) yields

$$\sum_{m=j+1}^{d} \xi^{d}_{k_1,..,k_m} = \sum_{m=j+1}^{d} 2^{d-m}\, \xi^{m-1}_{k_1,..,k_j,m}. \tag{A.19}$$

With (A.19), the double sum in (A.12) can be expressed as

$$\frac{-\pi}{2^{d+1}} \sum_{j=0}^{d-1} \sum_{m=j+1}^{d} 2^{d-m}\, \xi^{m-1}_{k_1,..,k_j,m}\, f_{j\,k_1,..,k_{j+1}} = \frac{-\pi}{2^{d+1}} \sum_{m=1}^{d} \sum_{j=0}^{m-1} 2^{d-m}\, \xi^{m-1}_{k_1,..,k_j}\, f_{j\,k_1,..,k_j,\,m}. \tag{A.20}$$

The function $f_{j\,k_1,..,k_j,\,m}$ is given by (A.10) with $k_{j+1}$ equal to $m$. Substituting (A.20) into (A.12) yields our final expression for the Neumann regularized vacuum energy

$$\tilde{E}_N = E_{0_N} + E_N(\Lambda) \tag{A.21}$$

where the finite part $E_{0_N}$ is given by

$$E_{0_N} = \frac{-\pi}{2^{d+1}} \sum_{m=1}^{d} \sum_{j=0}^{m-1} 2^{d-m}\, \xi^{m-1}_{k_1,..,k_j}\, \frac{L_{k_1}\ldots L_{k_j}}{(L_m)^{j+1}} \left(\Gamma(\tfrac{j+2}{2})\, \pi^{\frac{-j-4}{2}}\, \zeta(j+2) + R_{N_j}\right). \tag{A.22}$$

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The function $R_{N_j}$ is given by (A.4) with $L_1 \to L_{k_1}$, $L_{j+1} \to L_{k_{j+1}} = L_m$:

$$R_{N_j} = \sum_{n=1}^{\infty} {\sum_{\substack{\ell_i=-\infty \\ i=1,\dots,j}}^{\infty}}' \frac{2\,n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\big(2\pi\,n\,\sqrt{(\ell_1 \frac{L_{k_1}}{L_m})^2 + \cdots + (\ell_j \frac{L_{k_j}}{L_m})^2}\,\big)}{\Big[(\ell_1\frac{L_{k_1}}{L_m})^2 + \cdots + (\ell_j\frac{L_{k_j}}{L_m})^2\Big]^{\frac{j+1}{4}}}\,. \tag{A.23}$$

The Dirichlet case is obtained by substituting (A.16) in (A.11) yielding

$$\tilde{E}_D = \frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} (-1)^{d+j}\, \xi_{k_1,..,k_j}^{d-1}\, f_{j\,_{k_1,..,k_j,d}} + E_D(\Lambda)\,. \tag{A.24}$$

The function $f_{j\,_{k_1,..,k_j,d}}$ is obtained by setting $k_{j+1}$ equal to $d$ in (A.10) yielding the Dirichlet regularized vacuum energy

$$\tilde{E}_D = E_{0_D} + E_D(\Lambda) \tag{A.25}$$

where the finite part $E_{0_D}$ is given by

$$E_{0_D} = \frac{\pi}{2^{d+1}} \sum_{j=0}^{d-1} (-1)^{d+j}\, \xi_{k_1,..,k_j}^{d-1}\, \frac{L_{k_1}\dots L_{k_j}}{(L_d)^{j+1}} \Big(\Gamma(\tfrac{j+2}{2})\, \pi^{\frac{-j-4}{2}}\, \zeta(j+2) + R_{D_j}\Big)\,. \tag{A.26}$$

The function $R_{D_j}$ is given by (A.4) with $L_1 \to L_{k_1}$, $L_{j+1} \to L_{k_{j+1}} = L_d$:

$$R_{D_j} = \sum_{n=1}^{\infty} {\sum_{\substack{\ell_i=-\infty \\ i=1,\dots,j}}^{\infty}}' \frac{2\,n^{\frac{j+1}{2}}}{\pi} \frac{K_{\frac{j+1}{2}}\big(2\pi\,n\,\sqrt{(\ell_1 \frac{L_{k_1}}{L_d})^2 + \cdots + (\ell_j \frac{L_{k_j}}{L_d})^2}\,\big)}{\Big[(\ell_1\frac{L_{k_1}}{L_d})^2 + \cdots + (\ell_j\frac{L_{k_j}}{L_d})^2\Big]^{\frac{j+1}{4}}}\,. \tag{A.27}$$

For $j = 0$, $R_{N_j}$ and $R_{D_j}$ are defined to be zero and $\xi_{k_1,..,k_j}$ and $L_{k_j}$ are defined to be unity.

The final expressions for the regularized vacuum energy are (A.21) for the Neumann case and (A.25) for the Dirichlet case with the cut-off dependent parts $E_D(\Lambda)$ and $E_N(\Lambda)$ given by (A.13) and (A.14) and the finite parts $E_{0_N}$ and $E_{0_D}$ given by (A.22) and (A.26) respectively.

# B    Alternative expressions for the $d$-dimensional Casimir piston

In section 3 we derived expressions for the Casimir force on the piston. In this appendix we derive alternative expressions. This is accomplished by labeling the lengths $L_i$ in region I differently compared to section 3. The Casimir energy is invariant under permutation of lengths so a different labeling does not alter the value of the Casimir energy. However, the different labeling leads to an expression with a different form.

In section 3 we labeled the lengths $L_i$ in region I as $L_1=a_1, L_2=a_2,\ldots,L_{d-1}=a_{d-1}$ with $L_d$ equal to the plate separation $a$. We now label $L_i$ such that $L_1 = a, L_2 = a_1, L_3 = a_2,\ldots,L_d = a_{d-1}$ so that $L_1$ is equal to the plate separation. We do not change the labeling in region II (i.e. $L_1 = s - a, L_2 = a_1, L_3 = a_2$, etc.) so that we only need to obtain new formulas for region I.

The expressions for the finite part of the Casimir energy are given by (3.14) and (3.16) for Neumann and Dirichlet respectively. We only keep the $a$-dependent terms and for region I this means keeping only those terms with $L_1 = a$. For Neumann, (3.14) divides into two sums: the $j = 0$, $m = 1$ term where $L_m = L_1 = a$ appears in the denominator and all other terms where $L_1$ appears in the numerator (this occurs when $k_1 = 1$ so that $L_{k_1} = a$ for $j > 0$ with the other lengths given by $L_{k_j} = a_{k_j-1}$). This yields (with "alt" as superscript for "alternative")

$$E_{0_{N_I}}^{alt}(a) = -\frac{\pi}{2^{d+1}}\sum_{m=1}^{d}\sum_{j=0}^{m-1} 2^{d-m}\,\xi_{k_1,..,k_j}^{m-1}\,\frac{L_{k_1}\ldots L_{k_j}}{(L_m)^{j+1}}\left(\Gamma(\tfrac{j+2}{2})\,\pi^{\frac{-j-4}{2}}\,\zeta(j+2) + R_{N_j}\right)$$

$$= -\frac{\pi}{24\,a} - \frac{\pi}{2^{d+1}}\sum_{m=2}^{d}\sum_{j=1}^{m-1} 2^{d-m}\,\xi_{1,k_2,..,k_j}^{m-1}\,\frac{a\,a_{k_2-1}\ldots a_{k_j-1}}{(a_{m-1})^{j+1}}\left(\Gamma(\tfrac{j+2}{2})\,\pi^{\frac{-j-4}{2}}\,\zeta(j+2) + R_{I_{N_j}}^{alt}\right)$$

(B.1)

with

$$R_{I_{N_j}}^{alt} = \sum_{n=1}^{\infty}\,{\sum_{\substack{\ell_i=-\infty \\ i=1,\ldots,j}}^{\infty}}' \frac{2\,n^{\frac{j+1}{2}}}{\pi}\,\frac{K_{\frac{j+1}{2}}\left(2\pi\,n\,\sqrt{(\ell_1\frac{a}{a_{m-1}})^2 + \cdots + (\ell_j\frac{a_{k_j-1}}{a_{m-1}})^2}\,\right)}{\left[(\ell_1\frac{a}{a_{m-1}})^2 + \cdots + (\ell_j\frac{a_{k_j-1}}{a_{m-1}})^2\right]^{\frac{j+1}{4}}}$$

(B.2)

For Dirichlet given by (3.16), the case $j = 0$ does not yield any $L_1$ terms so that it can be dropped. $L_1 = a$ appears only in the numerator via $L_{k_1} = a$ when $k_1 = 1$ (with the other lengths given by $L_{k_j} = a_{k_j-1}$). We obtain

$$E_{0_{D_I}}^{alt}(a) = \frac{\pi}{2^{d+1}}\sum_{j=1}^{d-1}(-1)^{d+j}\,\xi_{1,..,k_j}^{d-1}\,\frac{a\,a_{k_2-1}\ldots a_{k_j-1}}{(a_{d-1})^{j+1}}\left(\Gamma(\tfrac{j+2}{2})\,\pi^{\frac{-j-4}{2}}\,\zeta(j+2) + R_{I_{D_j}}^{alt}\right)$$

(B.3)

with

$$R_{I_{D_j}}^{alt} = \sum_{n=1}^{\infty}\,{\sum_{\substack{\ell_i=-\infty \\ i=1,\ldots,j}}^{\infty}}' \frac{2\,n^{\frac{j+1}{2}}}{\pi}\,\frac{K_{\frac{j+1}{2}}\left(2\pi\,n\,\sqrt{(\ell_1\frac{a}{a_{d-1}})^2 + \cdots + (\ell_j\frac{a_{k_j-1}}{a_{d-1}})^2}\,\right)}{\left[(\ell_1\frac{a}{a_{d-1}})^2 + \cdots + (\ell_j\frac{a_{k_j-1}}{a_{d-1}})^2\right]^{\frac{j+1}{4}}}\,.$$

(B.4)

The alternative expression for the Casimir force in region I for Neumann is

$$F_{N_I}^{alt} = -\frac{\partial}{\partial a} E_{0_{N_I}}^{alt}(a)$$

$$= -\frac{\pi}{24\,a^2} + \frac{\pi}{2^{d+1}} \sum_{m=2}^{d} \sum_{j=1}^{m-1} 2^{d-m}\, \xi_{1,k_2,..,k_j}^{m-1}\, \frac{a_{k_2-1}\cdots a_{k_j-1}}{(a_{m-1})^{j+1}}\, \Gamma(\tfrac{j+2}{2})\, \pi^{\frac{-j-4}{2}}\, \zeta(j+2) - \frac{\partial R_{I_N}^{alt}}{\partial a}$$

(B.5)

where

$$R_{I_N}^{alt} = -\frac{\pi}{2^{d+1}} \sum_{m=2}^{d} \sum_{j=1}^{m-1} 2^{d-m}\, \xi_{1,k_2,..,k_j}^{m-1}\, \frac{a\, a_{k_2-1}\cdots a_{k_j-1}}{(a_{m-1})^{j+1}}\, R_{I_{N_j}}^{alt}\,.$$

(B.6)

The corresponding alternative expression for Dirichlet in region I is

$$F_{D_I}^{alt} = -\frac{\partial}{\partial a} E_{0_{D_I}}^{alt}(a)$$

$$= -\frac{\pi}{2^{d+1}} \sum_{j=1}^{d-1} (-1)^{d+j}\, \xi_{1,..,k_j}^{d-1}\, \frac{a_{k_2-1}\cdots a_{k_j-1}}{(a_{d-1})^{j+1}}\, \Gamma(\tfrac{j+2}{2})\, \pi^{\frac{-j-4}{2}}\, \zeta(j+2) - \frac{\partial R_{I_D}^{alt}}{\partial a}$$

(B.7)

where

$$R_{I_D}^{alt} = \frac{\pi}{2^{d+1}} \sum_{j=1}^{d-1} (-1)^{d+j}\, \xi_{1,..,k_j}^{d-1}\, \frac{a\, a_{k_2-1}\cdots a_{k_j-1}}{(a_{d-1})^{j+1}}\, R_{I_{D_j}}^{alt}\,.$$

(B.8)

To obtain the Casimir force on the piston we need to add the contribution from region II: $F_{N_{II}}$ and $F_{D_{II}}$ given by (3.30) and (3.31) respectively. For Neumann we obtain

$$F_N^{alt} = F_{N_I}^{alt} + F_{N_{II}} = -\frac{\pi}{24\,a^2} - \frac{\partial R_{I_N}^{alt}}{\partial a} - \frac{\pi}{2^{d+1}} \sum_{m=3}^{d} \sum_{j=2}^{m-1} 2^{d-m}\, \xi_{1,k_2,..,k_j}^{m-1}\, \frac{a_{k_2-1}\cdots a_{k_j-1}}{(a_{m-1})^{j+1}}\, R_{II_{N_j}}(\ell_1=0)$$

(B.9)

where $R_{II_{N_j}}(\ell_1=0)$ is (3.23) evaluated at $\ell_1=0$. The above can be reduced to a more compact expression by noticing that the $\ell_1 = 0$ contribution to $-\partial R_{I_N}^{alt}/\partial a$ cancels out with the last term in (B.9). The alternative expression for the Casimir force on the piston for Neumann boundary conditions reduces to

$$F_N^{alt} = -\frac{\pi}{24\,a^2} - \frac{\partial}{\partial a} R_{I_N}^{alt}(\ell_1 \neq 0)\,.$$

(B.10)

To evaluate (B.10) we exclude $\ell_1 = 0$ in (B.6) so that $R_{I_{N_j}}^{alt}$ given by (B.2) is evaluated without including $\ell_1 = 0$ i.e.

$$R_{I_{N_j}}^{alt}(\ell_1 \neq 0) = \sum_{n=1}^{\infty} \sum_{\ell_1=1}^{\infty} \sum_{\substack{\ell_i=-\infty \\ i=2,...,j}}^{\infty} \frac{4\, n^{\frac{j+1}{2}}}{\pi}\, \frac{K_{\frac{j+1}{2}}\left(2\pi n\, \sqrt{(\ell_1 \frac{a}{a_{m-1}})^2 + \cdots + (\ell_j \frac{a_{k_j-1}}{a_{m-1}})^2}\,\right)}{\left[(\ell_1 \frac{a}{a_{m-1}})^2 + \cdots + (\ell_j \frac{a_{k_j-1}}{a_{m-1}})^2\right]^{\frac{j+1}{4}}}\,.$$

(B.11)

Compared to (B.2), there is no longer a prime on the sum over $\ell_i$ and $i$ starts with the integer 2 instead of 1. For Dirichlet one obtains

$$F_D^{alt} = F_{D_I}^{alt} + F_{D_{II}} = -\frac{\partial R_{I_D}^{alt}}{\partial a} + \frac{\pi}{2^{d+1}} \sum_{j=2}^{d-1} (-1)^{d+j} \, \xi_{1,k_2,..,k_j}^{d-1} \, \frac{a_{k_2-1} \dots a_{k_j-1}}{(a_{d-1})^{j+1}} \, R_{II_{D_j}}(\ell_1=0) \quad \text{(B.12)}$$

where $R_{II_{D_j}}(\ell_1=0)$ is (3.25) evaluated at $\ell_1 = 0$. The above can also be reduced to a more compact expression since the $\ell_1=0$ contribution of $-\partial R_{I_D}^{alt}/\partial a$ cancels out with the last term in (B.12). The alternative expression for the Casimir force on the piston for Dirichlet boundary conditions reduces to the simple expression

$$F_D^{alt} = -\frac{\partial R_{I_D}^{alt}}{\partial a}(\ell_1 \neq 0) \,. \quad \text{(B.13)}$$

To evaluate (B.13) we exclude $\ell_1=0$ in (B.8) so that $R_{I_{D_j}}^{alt}$ given by (B.4) is evaluated without $\ell_1=0$ i.e.

$$R_{I_{D_j}}^{alt}(\ell_1 \neq 0) = \sum_{n=1}^{\infty} \sum_{\ell_1=1}^{\infty} \sum_{\substack{\ell_i=-\infty \\ i=2,\dots,j}}^{\infty} \frac{4 \, n^{\frac{j+1}{2}}}{\pi} \, \frac{K_{\frac{j+1}{2}}\left( 2\pi \, n \, \sqrt{(\ell_1 \frac{a}{a_{d-1}})^2 + \dots + (\ell_j \frac{a_{k_j-1}}{a_{d-1}})^2} \; \right)}{\left[ (\ell_1 \frac{a}{a_{d-1}})^2 + \dots + (\ell_j \frac{a_{k_j-1}}{a_{d-1}})^2 \right]^{\frac{j+1}{4}}} \,. \quad \text{(B.14)}$$

Our alternative expression for the Casimir force on the piston for the Neumann case is $F_N^{alt}$ which is given by (B.10) together with (B.6) and (B.11). For the Dirichlet case the alternative expression is $F_D^{alt}$ which is given by (B.13) together with (B.8) and (B.14). It is now trivial to see that in the limit as the plate separation $a$ tends to infinity that the Casimir force on the piston is zero since the modified Bessel functions and their derivatives that appear in $F_N^{alt}$ and $F_D^{alt}$ via (B.11) and (B.14) decrease exponentially fast to zero as $a$ tends to infinity.

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